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Gleason's Theorem and Completeness Criteria

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We give some applications of Gleason's theorem to completeness criteria of inner product spaces using different families of subspaces, measures on them, and frame functions. Some open criteria problems are formulated.

1. INTRODUCTION

In his investigation of the mathematical foundations of quantum mechanics, Mackey (1963) posed the following problem: *Describe the set of all states on the quantum logic $L(H)$ for a separable real or complex Hilbert space H.*

We recall that a state on $L(H)$ (H is not necessarily separable) is a mapping $m: L(H) \to [0, 1]$ such that

$$m(H) = 1 \tag{1}$$

$$m\left(\bigoplus_{i=1}^{\infty} M_i\right) = \sum_{i=1}^{\infty} m(M_i) \tag{2}$$

[here by $\bigoplus_{t\in T} M_t$ we shall mean the join of a family of mutually orthogonal subspaces $\{M_t: t\in T\}$ of $L(H)$]. Then

$$m_x(M) = \|P_M x\|^2, \qquad M\in L(H) \tag{3}$$

where P_M is the orthoprojector from H onto M, is a state.

If $\{x_i\}$ is a system of unit vectors and $\{\lambda_i\}$ is a system of positive numbers such that $\sum_i \lambda_i = 1$, then

$$m(M) = \sum_i \lambda_i m_{x_i}(M), \qquad M\in L(H) \tag{4}$$

is also a state.

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The latter is equivalent to the following:

$$m_T(M) = \mathrm{tr}(TP_M), \qquad M \in L(H) \tag{5}$$

Gleason (1957) published the answer to Mackey's problem in 1957:

Theorem 1.1 (Gleason's theorem). If H is a separable, real or complex Hilbert space, $\dim H \neq 2$, then for any state m on $L(H)$ there exists a unique positive, Hermitian trace operator T on H with $\mathrm{tr}\, T = 1$ such that

$$m(M) = \mathrm{tr}(TP_M), \qquad M \in L(H) \tag{6}$$

This theorem is a cornerstone for our investigation of completeness criteria of inner product spaces; for other applications of Gleason's theorem see, e.g., Dvurečenskij (1993).

If we omit the completeness assumption on H, we obtain inner product spaces which possess Hilbert spaces as a proper subclass. Recently different families of closed subspaces of inner product spaces have been used as axiomatic models. Therefore it is of great importance for quantum mechanical models to know completeness criteria of inner product spaces.

There are many interesting characterizations of Hilbert spaces using algebraic or topological properties (Amemiya and Araki, 1966/1967; Gross and Keller, 1977; Cattaneo and Marino, 1986; Holland, 1969; Dvurečenskij, 1988; Gudder, 1974, 1975; Gudder and Holland, 1975) as well as measure-theoretical ones, which started with Hamhalter and Pták (1987). Let S be a real or complex inner product space with an inner product $(\cdot, \cdot)$. We recall that for $M \subseteq S$, $M \neq \emptyset$, by $M^\perp$ we mean the set of all $x \in S$ such that $(x, y) = 0$ for each $y \in M$. We introduce the following families of closed subspaces that show quite different properties from the ordering point of view:

$$W(S) = \{M \subseteq S\colon M \text{ is a closed subspace of } S\}$$

$$F(S) = \{M \subseteq S\colon M^{\perp\perp} = M\}$$

$$D(S) = \{M \subseteq S\colon \exists \text{ONS } \{u_i\}, M = \{u_i\}^{\perp\perp}\}$$

$$R(S) = \{M \subseteq S\colon M = \{u_i\}^{\perp\perp}, \forall \text{ MONS } \{u_i\} \text{ in } M\}$$

$$V(S) = \{M \subseteq S\colon M = \{u_i\}^{\perp\perp}, M^\perp = \{v_j\}^{\perp\perp}, \forall \text{ MONSs } \{u_i\} \text{ in } M, \text{ and } \{v_j\} \text{ in } M^\perp\}$$

$$E(S) = \{M \subseteq S\colon M + M^\perp = S\}$$

$$C(S) = \{M \subseteq S\colon \dim M < \infty \text{ or } \dim M^\perp < \infty\}$$

$$P(S) = \{M \subseteq S\colon \dim M < \infty\}$$

It is easy to see that

$$P(S) \subseteq C(S) \subseteq E(S) \subseteq V(S) \subseteq R(S) \subseteq D(S) \subseteq F(S) \subseteq W(S) \qquad (7)$$

and, as we shall see, many of these inclusions can be proper.

A mapping m from $K(S)$, where K is the capital from the set $\{C, E, V, R, D, F, W\}$, into the real line $\mathbf{R}$ such that

$$m\left(\bigoplus_{i \in I} M_i\right) = \sum_{i \in I} m(M_i) \qquad (8)$$

and for $K = W$ we add the condition

$$m(M \vee M^{\perp}) = m(S), \qquad M \in W(S) \qquad (9)$$

whenever $\{M_i : i \in I\}$ is a system of mutually orthogonal subspaces of $K(S)$ for which the join $\bigoplus_{i \in I} M_i$ exists in $K(S)$, is said to be a *charge*, a *signed measure*, or a *completely additive signed measure* if (9) holds for any finite, countable, or arbitrary index set I. If m attains only positive values, we say that m is a *finitely additive measure*, *measure*, or *completely additive measure*, respectively, according to the cardinality of I. A finitely additive measure m such that $m(S) = 1$ is said to be *Jordan* if it can be represented as a difference of two positive finitely additive measures.

2. ALGEBRAIC COMPLETENESS CRITERIA

The system $F(S)$ is a complete orthocomplemented lattice which is not orthomodular, in general. The orthomodularity means: if $M \subseteq N$, $M, N \in F(S)$, then

$$N = M \vee (N \wedge M^{\perp}) \qquad (10)$$

An important result of Amemiya and Araki (1966/1967) gives the first algebraic characterization:

Theorem 2.1 (Amemiya–Araki). An inner product space S is complete if and only if $F(S)$ is an orthomodular lattice.

The system of all splitting subspaces of S, $E(S)$, is an orthomodular poset (OMP for short).

The OMP $E(S)$ has been used to the completeness characterization of inner product spaces by (1) Gross and Keller (1977): S is complete iff $E(S)$ is a complete lattice; and (2) Cattaneo and Marino (1986): S is complete iff $E(S)$ is a σ-lattice. The author (Dvurečenskij, 1988) has weakened these conditions showing that S is complete iff $E(S)$ is a quantum logic, that is, a σ-OMP:

Theorem 2.2 (Dvurečenskij, 1988). An inner product space S is complete if and only if for any sequence of orthonormal vectors $\{x_n\}_{n=1}^{\infty}$ in S we have $\{x_n\}^{\perp\perp} \in E(S)$. In particular, S is complete if and only if $E(S)$ is a σ-OMP.

Now we show that Theorem 2.2 can be weakened as follows (oral communication by Dr. P. Pták). We say that $E(S)$ has (1) the *subsequential interpolation property* if for any sequence of orthonormal vectors $\{e_n\}$ in S and any of its subsequences $\{e_{n_i}\}$ there is a splitting subspace M of S such that $e_{n_i} \in M$ for any i, and $e_k \perp M$ for any $e_k \in \{e_n\} - \{e_{n_i}\}$; (2) the *strong subsequential completeness property* if for any sequence of linearly independent vectors $\{e_n\}$ in S there is its subsequence $\{e_{n_i}\}$ such that $\{e_{n_i}\}^{\perp\perp} \in E(S)$. These notions play an important role in Brook–Jewett theorem formulations.[2]

Theorem 2.3. Let S be an inner product space. The following statements are equivalent:

1. S is complete.
2. $E(S)$ has the subsequential interpolation property.
3. $E(S)$ has the strong subsequential completeness property.

The system $W(S)$ is a complete lattice with orthocomplementation such that

$$M \vee M^{\perp} \subseteq S, \qquad M \subseteq M^{\perp\perp} \tag{11}$$

when proper inclusions in (11) are not excluded, as the following example shows:

Example 2.4. Let $S = l_f \subset l_2$ be the set of all sequences from l_2 which are nonzero only for finitely many numbers of its terms. Let $M = \{\{c_k\} \in S: \sum_k c_k/k = 0\}$. Then

$$M \in W(S), \qquad M^{\perp} = \{0\}, \qquad M = M \vee M^{\perp} \neq S, \qquad M \neq M^{\perp\perp} = S$$

We recall the invalidity of the "excluded middle law" $M \vee M^{\perp} = S$, $M \in W(S)$, can have some applications for using $W(S)$ in fuzzy set models for quantum mechanics.

To present the criterion of Holland (1969), we introduce the following notions: Two elements a and b of a lattice L are said to form a *modular pair*

[2]We recall that, in general (D'Andrea and de Lucia, 1991), an OMP L has (1) the subsequential interpolation property if for any orthogonal sequence $\{a_n\}$ in L and any of its subsequences $\{a_{n_i}\}$ there is an element $b \in L$ such that $a_{n_i} \leq b$ for any i, and $a_k \perp b$ for any $a_k \in \{a_n\} - \{a_{n_i}\}$; and (2) the subsequential completeness property if for any orthogonal sequence $\{a_n\}$ in L there is a subsequence $\{a_{n_i}\}$ such that $\bigvee_i a_{n_i} \in L$.

when $(x \vee a) \wedge b = x \vee (a \wedge b)$ holds for all $x \leq b$ and are said to form a *dual-modulator pair* when $(x \wedge a) \vee b = x \wedge (a \vee b)$ holds for all $b \leq x$.

Theorem 2.5 (Holland). An inner product space S is complete if and only if every modular pair in $W(S)$ is dual-modular.

3. MEASURE-THEORETIC COMPLETENESS CRITERIA

First we introduce some notations. We denote by $P(S)$ and $P_1(S)$ the sets of all finite-dimensional and all one-dimensional subspaces of S, respectively. We say that a charge m on $K(S)$ is (i) *bounded* if $\sup\{|m(M)|: M \in K(S)\} < \infty$ [*semibounded* if $\inf\{m(M): M \in K(S)\} > -\infty$]; (ii) $P(S)$-*bounded* if $\sup\{|m(M)|: M \in P(S)\} < \infty$ [$P(S)$-*semibounded* if $\inf\{m(M): M \in P(S)\} > -\infty$]; (iii) $P_1(S)$-*bounded* if $\sup\{|m(M)|: M \in P_1(S)\} < \infty$ [$P_1(S)$-*semibounded* if $\inf\{m(M): M \in P_1(S)\} > -\infty$].

We recall that if m is a Jordan charge on $K(S)$, then m is bounded in all the above senses.

It is easy to see that if m is a charge on $W(S)$, then for any $M \in W(S)$ we have from (9) that

$$m(M) = m(M^{\perp\perp}) \tag{12}$$

The crucial role for our purposes will be played by the following lemma; its second part for separable S and finitely additive states has been proved in Hamhalter and Pták (1987). Here we present a more general variant.

Lemma 3.1. (1) For any $P_1(S)$-bounded charge m on $F(S)$ or $E(S)$, $\dim S \neq 2$, there exists a unique Hermitian operator $T = T_m: \bar{S} \to \bar{S}$ such that

$$m(\mathrm{sp}(x)) = (Tx, x), \qquad x = \mathcal{S}(S) \tag{13}$$

(2) Let v be a unit vector in the completeness $\bar{S}$ of S, $\dim S \neq 2$. Then for any $\epsilon > 0$ and any $K > 0$, there exists a $\delta > 0$ such that the following statement holds: If $w \in S$ is a unit vector such that $\|v - w\| < \delta$, then for any $P(S)$-bounded charge m such that the norm of $T = T_m$ is less than K, and for each finite-dimensional A satisfying the property $v \perp A$, we have the inequality

$$\|m(A \vee \mathrm{sp}(w)) - m(A) - m(\mathrm{sp}(w))\| < \epsilon \tag{14}$$

Now we present the first measure-theoretic completeness criterion:

Theorem 3.2 (Dvurečenskij and Pulmannová, 1989). An inner product space S is complete if and only if $E(S)$ possesses at least one nonzero completely additive signed measure m.

Theorem 3.3. An inner product space S is complete if and only if $K(S)$, where K is a capital letter from $\{E, V, R, D, F, W\}$, possesses at least one nonzero completely additive signed measure.

The application of Theorem 3.2 gives the following non-measure-theoretic criterion.

Let $\mathscr{S}(S)$ be a unit sphere in S, that is, $\mathscr{S}(S) = \{x \in S: \|x\| = 1\}$. A mapping $f: \mathscr{S}(S) \to \mathbf{R}$ such that there is a finite constant W, called the *weight* of f such that, for any MONS $\{x_i\}$ is S, we have

$$\sum_i f(x_i) = W$$

is said to be a *frame function* on S. It is clear that $f(\lambda x) = f(x)$ for any scalar $\lambda \in \mathbf{D}$, $|\lambda| = 1$.

Theorem 3.4. An inner product space S is complete if and only if S possesses at least one nonzero frame function.

The criterion of Theorem 3.4 has an interesting consequence for the criterion of Gudder and Holland (1975; Gudder, 1974, 1975), which says:

Theorem 3.5. An inner product space S is complete if and only if any MONS in S is an ONB in S, i.e.,

$$\forall x \in S, \quad \forall \text{MONS } \{x_i\} \text{ of } S, \qquad x = \sum_i (x_i, x)x_i \tag{15}$$

We show now that (15) can be remarkably weakened as follows:

Theorem 3.6 (Dvurečenskij, 1989*b*, 1990*a*). An inner product space S is complete if and only if

$$\exists 0 \neq x \in S \; (x \in \bar{S}) \quad \forall \text{MONS } \{x_i\} \text{ of } S, \qquad x = \sum_i (x_i, x)x_i \tag{16}$$

Proof. If we put $f(u) = |(u, x)|^2$, $u \in \mathscr{S}(S)$, then by (16), f is a nonzero frame function on S, and applying Theorem 3.4, S becomes complete. QED

4. GLEASON'S THEOREM AND REGULAR CHARGES

We say that a charge m on $E(S)$ $[F(S)]$ is $P(S)$-*regular* if, given $M \in E(S)$ $[M \in F(S)]$ and given $\epsilon > 0$ there is a finite-dimensional subspace N of M such that

$$|m(M \cap N^{\perp}) < \epsilon \tag{17}$$

Proposition 4.1. Any $E(S)$, $\dim S > 1$, possesses unbounded charges.

Proof. Take the additive discontinuous functional ϕ on $\mathbf{R}$. Now let $T \neq kI$ be a Hermitian trace operator on $\bar{S}$, where k is a nonzero real constant and I is the identity on $\bar{S}$. The mapping m: $E(S) \to \mathbf{R}$ defined as

$$m(M) = \phi(\mathrm{tr}(TP_{\bar{M}})), \qquad M \in E(S) \tag{18}$$

is an unbounded charge on $E(S)$. QED

The following gives a generalization of the Aarnes (1970) decomposition and Yosida–Hewitt-type decomposition (Yosida and Hewitt, 1952; Rüttimann, 1990).

Theorem 4.2. Any Jordan charge m on $E(S)$, $\dim S \neq 2$, can be uniquely decomposed as a sum $m = m_1 + m_2$, where m_1 is a $P(S)$-regular charge, and m_2 is a Jordan charge vanishing on all finite-dimensional subspaces of S.

As a corollary we present Gleason's analog for finitely additive measures on $E(S)$:

Theorem 4.3. A Jordan charge m on $E(S)$, $\dim S \neq 2$, is $P(S)$-regular if and only if there is a Hermitian operator $T \in \mathrm{Tr}(\bar{S})$ such that

$$m(M) = \mathrm{tr}(TP_{\bar{M}}), \qquad M \in E(S) \tag{19}$$

We recall that in $E(S)$ we have plenty of regular states [they are given by formula (19)] even for any incomplete S. For $F(S)$ the situation is quite different:

Theorem 4.4. An inner product space S is complete if and only if $F(S)$ or $W(S)$ possesses at least one nonzero Jordan $P(S)$-regular charge.

As an interesting corollary of application of Gleason's theorem for finite-dimensional cases (and in incomplete S we do only that) we present the following result:

Proposition 4.5. Let $\dim S \geq 3$; then on $K(S)$, where $K \in \{D, F, W\}$, there is no two-valued state.

Via $M \mapsto \bar{M}$, $M \in E(S)$, $E(S)$ can be embedded injectively into $E(\bar{S})$. We present six open problems:

Problem 4.6. (1) does $F(S)$ or $W(S)$ possess a finitely additive nonzero Jordan charge which is not $P(S)$-regular?

(2) Is it possible to extend any finitely additive state on $E(S)$ vanishing on $P(S)$ to a finitely additive state on $E(\bar{S})$?

(3) Does the Nikodým convergence theorem hold for $P(S)$-regular Jordan charges on $E(S)$?

(4) Is S complete if for any pair of MONSs $\{e_i\}$ and $\{f_i\}$ in S there is a unitary operator $U: S \to S$ such that $Ue_i = f_i$ for any i?

(5) Is S complete if there exist $x \in \mathscr{S}(S)$ and a positive constant $A > 0$ such that $A\|x\|^2 \leq \sum_i |(e_i, x)|^2$ holds for any MONS $\{e_i\}$ in S?

(6) Is S complete if for any $P(S)$-regular Jordan charge m on $E(S)$ there exists a Hahn decomposition, i.e., an element $M \in E(S)$ such that $m(N) \leq 0$ if $N \subseteq M$ and $m(N) \geq 0$ if $N \subseteq M^\perp$?

The author has found a positive solution to (5).

It is clear that if x is a unit vector in $\bar{S}$, then the mapping $m_x: E(S) \to [0, 1]$ such that $m_x(M) = \|P_{\bar{M}}x\|^2$, $M \in E(S)$, is a $P(S)$-regular state on $E(S)$. For complete S, m_x has such important properties as the *σ-Jauch–Piron property* (*σ-weak Jauch–Piron property*) if for any sequence of splitting subspaces of S (any sequence of one-dimensional subspaces of S), $\{M_n\}$, with $m(M_n) = 0$ for any $n \geq 1$ there is an $M \in E(S)$ such that $M_n \subseteq M$ for each $n \geq 1$, and $m(M) = 0$. m_x also has a support, i.e., an element $M \in E(S)$ such that $m_x(N) = 0$ iff $N \perp M$. For incomplete S the situation is quite different:

Theorem 4.7. If $\dim S \neq 2$, the following statements are equivalent:

(1) S is complete.
(2) Any $P(S)$-regular finitely additive state on $E(S)$ has the σ-Jauch–Piron property.
(3) Any $P(S)$-regular finitely additive state on $E(S)$ has the σ-weak Jauch–Piron property.
(4) Any m_x, $x \in \mathscr{S}(\bar{S})$, on $E(S)$ has the σ-weak Jauch–Piron property.
(5) Any m_x, $x \in \mathscr{S}(\bar{S})$, on $E(S)$ has a support.

5. COMPLETENESS CRITERIA SURVEY

For the reader's convenience we summarize the above completeness criteria for inner product spaces; for more details see Dvurečenskij (1993).

Theorem 5.1. Let S be an inner product space. The following statements are equivalent:

1. S is complete.
2. $E(S) = W(S)$ (Gudder, 1974).
3. $F(S) = W(S)$ (Gudder, 1974).
4. For any proper closed subspaces M of S, $M^\perp \neq \{0\}$ (Gudder, 1974).

5. If f is a continuous linear functional on S, there exists $y \in S$ such that $f(x) = (x, y)$ for all $x \in S$ (Gudder, 1974).
6. For any nonzero continuous linear functional f on S, $(\mathrm{Ker}\, f)^{\perp} \neq \{\mathbf{0}\}$.
7. For any continuous linear functional f on S, $\mathrm{Ker}\, f \in E(S)$.
8. $F(S)$ is orthomodular (Amemiya and Araki, 1966/1967, Theorem 2.1).
9. $E(S) = F(S)$.
10. $E(S)$ is a complete lattice (Gross and Keller, 1977).
11. $E(S)$ is a σ-lattice (Cattaneo and Marino, 1986).
12. $E(S)$ is a σ-orthoposet (=quantum logic) (Dvurečenskij, 1988, Theorem 2.2).
13. $E(S)$ possesses the join of any sequence of mutually orthogonal one-dimensional subspaces of S (Dvurečenskij, 1988, Theorem 2.2).
14. $E(S)$ has the subsequential interpolation property.
15. $E(S)$ has the strong subsequential completeness property.
16. Any modular pair in $W(S)$ is dual-modular (Holland, 1969).
17. $D(S)$ is an OML (Canetti and Marino, 1988).
18. $R(S) = F(S)$ (Cattaneo *et al.*, 1987, Theorem 2.2).
19. $D(S) = E(S)$ (Canetti and Marino, 1988).
20. $K(S)$, if $K \in \{C, E, V, R, D, F, W\}$, possesses at least one nonzero, completely additive signed measure (Dvurečenskij and Pulmanová, 1988, 1989).
21. S possesses at least one nonzero frame function (Dvurečenskij, 1989*b*, 1990*a*).
22. Every MONS in S is an ONB in S (Gudder and Holland, 1975, Theorem 3.5).
23. There exists a unit vector $y \in \bar{S}$ such that $y = \sum_i (y, x_i)x_i$ for any MONS $\{x_i\}$ in S (Dvurečenskij, 1989*b*, 1990*a*).
24. $F(S)$ [$W(S)$] possesses at least one Jordan $P(S)$-regular, nonzero charge (Dvurečenskij, 1991, n.d.).
25. $K(S)$, where $K \in \{E, V, R\}$ and dimension of S is a countable cardinal, possesses at least one nontrivial, strongly $P(S)$-regular, finitely additive measure (Dvurečenskij *et al.*, 1992; Dvurečenskij, 1990*b*).
26. $K(S)$, where $K \in \{D, F, W\}$, possesses at least one state having a support.
27. $K(S)$, if $K \in \{D, F, W\}$ and the dimension of S is a nonmeasurable cardinal, possesses at least one nontrivial signed measure (Dvurečenskij, 1989*a*,*b*).
28. $K(S)$, where $K \in \{F, W\}$ possesses at least one signed measure nonvanishing on $P(S)$ (Dvurečenskij, 1993*a*).

29. $K(S)$, where $K \in \{F, W\}$ and S is Dacey,[3] possesses at least one signed measure nonvanishing on $P(S)$ (Dvurečenskij, 1991).
30. $K(S)$, where $K \in \{F, W\}$ possesses at least one finitely additive state with a finite-dimensional support (Dvurečenskij, 1991).
31. $K(S)$, where $K \in \{F, W\}$, possesses at least one finitely additive state with a $P(S)$-regular support (Dvurečenskij, 1992*b*).
32. Any $P(S)$-regular state on $E(S)$ has a support in $E(S)$ (Dvurečenskij, 1992*b*).
33. For any sequence $\{x_i\}$ of orthonormal vectors in S and all positive numbers $\{\lambda_i\}$, with $\sum_i \lambda_i = 1$, the state $\sum_i \lambda_i m_{x_i}$ has a support in $E(S)$ (Dvurečenskij, 1992*b*).
34. For any infinite sequence $\{x_i\}$ of orthonormal vectors in S, the state $\sum_i m_{x_i}/2^i$ has a support in $E(S)$ (Dvurečenskij, 1992*b*).
35. There is a strong system $\mathscr{M}$ of states on $E(S)$, $\dim S \neq 2$, such that any counter on $\mathrm{Con}(\mathscr{M})$ is σ-expectational (Dvurečenskij, 1992*b*).
36. Any $P(S)$-regular finitely additive state on $E(S)$ has the σ-Jauch–Piron property (Dvurečenskij, 1993).
37. Any $P(S)$-regular finitely additive state on $E(S)$ has the σ-weak Jauch–Piron property (Dvurečenskij, 1993).
38. Any m_x, $x \in \mathscr{S}(\bar{S})$, has the σ-weak Jauch–Piron property (Dvurečenskij, 1993).

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[3]We say that an ONS $\{u_i\}$ in S is *Dacey* if $\{u_i\} = \{u_{i_j}\} \cup \{u_{i_k}\}$ and $\{u_{i_j}\} \cap \{u_{i_k}\} = \emptyset$ imply $\{u_{i_j}\}^\perp = \{u_{i_k}\}^{\perp\perp}$. If any MONS in S is Dacey, S is said to be a *Dacey inner product space.*

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